THE NEOCLASSICAL MODEL OF SOLOW AND SWAN WITH LOGISTIC POPULATION GROWTH

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Abstract. This paper is an attempt at studying the neoclassical Solow-Swan model within a framework where the change over time of the labor-force is given by the logistic population model. In the canonical Solow-Swan model, the growth rate of population is constant, yielding an exponential behavior of population size over time, which is clearly unrealistic and unsustainable in the very long-run. A more realistic approach would be to consider a logistic law for the population growth rate. In this framework, the model is proved to have a unique equilibrium (a node), which is globally asymptotically stable, and its solution is shown to have a closed-form expression via Hypergeometric functions.

1. Introduction

Most modern dynamic models of macroeconomics build on the framework described in Solow’s (1956) and Swan’s (1956) papers, two pathbreaking articles introducing the Solow-Swan model, or simply the Solow model for the more famous of the two economists. This model, which can be seen as the benchmark for what is now called the neoclassical theory of growth, aims to provide a theoretical framework for understanding world-wide growth of output and the persistence of geographical differences in per capita output. Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model, a model developed independently by Harrod (1939) and Domar (1946), which assumed fixed-coefficient production technologies that gave their models knife-edge equilibria, with the implausible implication that any deviation at all from equilibrium would cause the model to diverge further and further away from equilibrium. Solow criticized the Harrod-Domar model to analyze long-run problems with the usual short-run classical analysis and demonstrated why the Harrod-Domar model was not an attractive place to start. In his study, Solow took all the assumptions as given in Harrod-Domar model except the assumption of fixed proportions of input, and he extended this model by adding labor as a factor of production, by requiring diminishing returns to labor and capital separately, and constant returns to scale for both factors combined, and, finally, by introducing a time varying technology variable distinct from capital and labor. On the basis of these assumptions, in the long-run, output per capita converges to its steady state independently of initial conditions, the only potential sources of growth are sustained exogenous increases in primary factors, e.g. population growth, and exogenously given technological change. In addition, the long run growth rate is unaffected by the rate of savings, or investment. An increase of the saving rate has only a level effect, the steady state value of capital per worker grows, and not a growth effect. Thus, growth is exogenous in the sense that the behavior of economic agents does not alter the steady-state growth rate of the
economy. In Solow model, as simple as hypotheses go, the assumption of a constant labor growth rate is not a good approximation to reality. The main problem is that population grows exponentially, and so tends to infinity as time goes to infinity, which is clearly unrealistic. To remove the prediction of unbounded population size in the very long-run, Verhulst (1838) considered the hypothesis that any stable population would show a saturation level. By including an additional quadratic term with a negative coefficient in the exponential model, he wrote an alternative model, known as the logistic growth model, which reproduces this behavior. A natural question to be asked in Solow model is what the impact of changes in the population growth rate would be. Therefore, following Accinelli and Brida (2007), who studied the Ramsey model with logistic population growth, we introduce a modification in the neoclassical economic growth model by modeling population growth with the logistic population growth function. Notice that several studies support the hypothesis that the world’s population growth rate is decreasing and tends toward zero (see, for example, Day, 1996). Within this set-up, the model is shown to be described by a two dimensional dynamical system, which has a unique non-trivial steady state equilibrium (a node), whose solution is proved to be globally asymptotically stable and writable in closed-form via Hypergeometric functions.

2. The model

We consider a closed economy consisting of a single good, which is used either for consumption or investment, produced by labor $L_t$ and capital $K_t$ in a process described by a Cobb-Douglas production function

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1,$$

where $Y_t$ is the flow of output and $A_t$ is the level of technology. Physical capital tends to accumulate over time through investment. Gross investment $I_t$ has two components: net investment, defined as the variation in the stock of capital, $K_t$, and the loss by depreciation $D_t$, i.e. $I_t = K_t + D_t$. The rate of depreciation of physical capital, say $\delta$, is supposed to be constant, so that $D_t = \delta K_t$. Being the economy closed, the aggregate savings $S_t$ and investment are equal to each other every period, i.e. $S_t = I_t$. Additionally, let us suppose savings to evolve over time as a constant fraction $s$ of output, i.e. $S_t = sY_t$. Thus, we have

$$K_t = sY_t - \delta K_t. \tag{1}$$

Assuming full employment in the economy, so that employment and labor supply coincide, we can use in what follows total population as an input in the production function and write the technology in terms of per effective capita variables, i.e.

$$\frac{Y_t}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^\alpha \Rightarrow y_t = k_t^\alpha, \tag{2}$$

where $y_t = Y_t/A_t L_t$ and $k_t = K_t/A_t L_t$ denote income per unit of effective labor and the stock of capital per unit of effective labor, respectively. Taking derivatives with respect to time in the definition of $k_t$ yields

$$\dot{k}_t = \frac{d(K_t/A_t L_t)}{dt} = K_t \frac{A_t}{A_t L_t} - \left(\frac{A_t}{A_t} + \frac{L_t}{L_t}\right) k_t. \tag{3}$$
From equations (1), (2) and (3), we get
\[ k_t = sk_t^\alpha - \left( \delta + \frac{A_t}{A_t} + \frac{L_t}{L_t} \right) k_t. \]  \( (4) \)

Technological progress \( A_t \) grows at the constant rate \( g > 0 \), whereas, contrary to standard Solow-Swan model, population growth rate is not constant, but given by the logistic law of population growth, i.e.
\[ \frac{L_t}{L_t} = a - bL_t, \quad a > b > 0, \]  \( (5) \)

having assumed, for simplicity, \( L_0 = 1 \). Let us recall that, usually, standard economic growth theory considers that population grows exponentially. However, this hypothesis is realistic only for an initial period, but it cannot be valid indefinitely because population growing exponentially can be arbitrarily large. What is often observed instead is that as the population grows, some members interfere with each other in competition for some critical resource. That competition diminishes the growth rate, until the population ceases to grow. It seems reasonable that a good population model must therefore reproduce this behavior. The logistic population growth model, that was proposed by Verhulst (1838), is just such a model. With the inclusion of a logistic population growth law, from (4) and (5), we see that the economy of this modified Solow model is, thus, described by a system of two non-linear differential equations
\[
\begin{cases}
  \dot{k}_t = sk_t^\alpha - (\delta + g + a - bL_t) k_t, \\
  \dot{L}_t = (a - bL_t)L_t.
\end{cases}
\]  \( (6) \)

Given \( k_0 > 0 \), this Cauchy problem has a unique solution \((k_t, L_t)\), defined on \([0, \infty)\) (see Birkhoff and Rota, 1978).

3. Transitional dynamics

A steady state of the economy is defined as a situation in which the growth rate of per effective worker physical capital stock and the growth rate of labor are both equal to zero. Let \( k_* \), \( L_* \) denote the steady state equilibrium values of \( k_t \) and \( L_t \), respectively. In studying the steady states of our model, let us confine the analysis to interior steady states only, i.e. we will exclude the economically meaningless solutions such as \( k_* = 0 \) or \( L_* = 0 \).

**Proposition 1.** There exists a unique steady state equilibrium \((k_*, L_*)\), where
\[ k_* = \left( \frac{s}{\delta + g} \right)^{1/(1-\alpha)} \quad \text{and} \quad L_* = \frac{a}{b}. \]  \( (7) \)

**Proof.** Steady states are characterized by \( \dot{k}_t = 0 \) and \( \dot{L}_t = 0 \). From (6), we see that steady state levels of \( k_t \) and \( L_t \) are solutions to \( sk_t^\alpha - (\delta + g + a - bL_t) k_t = 0 \), \( L_t (a - bL_t) = 0 \), which give \( sk_t^{\alpha-1} = \delta + g, \ a - bL_t = 0 \), i.e. the statement. \( \Box \)

Outside the steady state the growth rate of the economy is not constant but, rather, it behaves according to (6), changing with the level of \( k_t \) and \( L_t \). To determine what the equilibrium path of our economy looks like we need to study the transitional dynamics of the dynamical system (6) starting from an arbitrary \( k_0 > 0 \). Of special interest is the answer to the question of whether the economy
will tend to this steady state starting from an arbitrary $k_0 > 0$, and how it will behave along the transition path.

**Theorem 1.** The steady state equilibrium is a stable node.

**Proof.** The local dynamic around $(k_*, L_*)$ is determined by the signs of the eigenvalues of the Jacobian matrix corresponding to its linearized system. This writes

\[
\begin{bmatrix}
  k_t \\
  L_t
\end{bmatrix}
= J^* \begin{bmatrix}
  k_t - k_* \\
  L_t - L_*
\end{bmatrix},
\]

where $J^* = \begin{bmatrix}
  J^*_{11} & J^*_{12} \\
  J^*_{21} & J^*_{22}
\end{bmatrix}$. \( (8) \)

$J^*$ is the Jacobian matrix of the system (6) evaluated at $(k_*, L_*)$, where, by definition, $J_{11}^* = \left( \partial k_t / \partial k_t \right)_{(k_*, L_*)}$, $J_{12}^* = \left( \partial k_t / \partial L_t \right)_{(k_*, L_*)}$, $J_{21}^* = \left( \partial L_t / \partial k_t \right)_{(k_*, L_*)}$, and $J_{22}^* = \left( \partial L_t / \partial L_t \right)_{(k_*, L_*)}$. These lead to

\[
J^* = \begin{bmatrix}
  -(1 - \alpha)\delta + g & bk_* \\
  0 & -a
\end{bmatrix}.
\]

The eigenvalues of this matrix are given by $\lambda_1 = -(1 - \alpha)(\delta + g)$ and $\lambda_2 = -a$. Since they are real and both negative, we can conclude that the equilibrium is a stable node, where the term node refers to the characteristic shape of the ensemble of orbits around the equilibrium. \( \square \)

**Remark 1.** The point $(k_*, L_*)$ is locally asymptotically stable in the sense that all solutions which start near it remain near the steady state for all time, and, furthermore, they tend towards $(k_*, L_*)$ as $t$ grows to infinity.

**Remark 2.** The general form of the solution to (8) is of the form

\[
\begin{align*}
  k_t &= k_* + c_1 e^{\lambda_1 t} + c_2 bk_* e^{\lambda_2 t}, & \text{if } \lambda_1 \neq \lambda_2, \\
  L_t &= L_* + c_2(\lambda_1 - \lambda_2)e^{\lambda_2 t},
\end{align*}
\]

and

\[
\begin{align*}
  k_t &= k_* + [c_1 + c_2(1 + t)]e^{\lambda_2 t}, & \text{if } \lambda_1 = \lambda_2, \\
  L_t &= L_* + (c_2/bk_*)(e^{\lambda_2 t},
\end{align*}
\]

where $c_1, c_2$ are constants determined from initial conditions.

After locating the fixed point and studying its stability, we next wish to ascertain whether (6) has any periodic orbits.

**Lemma 1.** A limit cycle cannot occur in this model.

**Proof.** From (6), we obtain

\[
\frac{\partial (k^{-1}_t L^{-1}_t k_{11})}{\partial k_t} + \frac{\partial (k^{-1}_t L^{-1}_t L_{11})}{\partial L_t} = -(1 - \alpha)sL^{-1}_t k^{-2}_t - bk^{-1}_t < 0.
\]

The result follows by applying the Bendixson-Dulac theorem with $B(k_1, L_t) = k^{-1}_t L^{-1}_t$ (see Boyce and DiPrima, 1995). \( \square \)

**Lemma 2.** Any solution to the system (6) is bounded, i.e. there exists a compact set $\Omega \subset \mathbb{R}_+^2$ such that $(k_t, L_t) \subset \Omega$ for all $t$.

**Proof.** The proof follows immediately from the Inada conditions. \( \square \)

From Lemmas 1 and 2, we can establish the following result.
Theorem 2. Any solution \((k_t, L_t)\) of the system converges to the steady state equilibrium \((k_*, L_*)\) as \(t \to \infty\).

Proof. By the Poincaré-Bendixson Theorem (see Boyce and DiPrima, 1995) every trajectory \((k_t, L_t)\) must become unbounded, or converge to a limit cycle, or converge to a steady state equilibrium. Boundedness is shown in Lemma 1 and limit cycles are ruled out in Lemma 2.

Remark 3. All trajectories, no matter where the initial point is, will be drawn to the steady state, i.e. the point \((k_*, L_*)\) is globally asymptotically stable in \(\mathbb{R}^2_+\) for the system (6).

4. The duration of the transition to steady state

We want to provide a quantitative assessment of the speed of transitional dynamics, i.e. to have an idea of how fast the economy approaches the steady state. It is important to know the speed of transitional dynamics since if convergence is rapid, we can focus on steady state behavior, because most economies would typically be closed to their steady state. Conversely, if convergence is slow, economies would be far from their steady state, and, hence, their growth experiences would be dominated by the transitional dynamics. The negative eigenvalues are the analogous to the convergence coefficient in the standard Solow growth model. The speed of convergence of the system is the absolute value of the highest negative eigenvalue. From Remark 2, we see that the speed of population depends on the eigenvalue \(\lambda_2\), whether the speed of capital depends on the eigenvalues \(\lambda_1\) and \(\lambda_2\). Consequently, if \(\lambda_2 < \lambda_1\), then the speed of convergence of \(L\) is faster than that of \(k\), while if \(\lambda_1 < \lambda_2\), then all variables converge at the same speed \(-\lambda_2\). Notice that, contrary to the Solow model, where the stable manifold is a one-dimensional locus, so that the speed of adjustment is parameterized unambiguously by the magnitude of a unique stable eigenvalue, now the speed of convergence may be parameterized by the magnitude of two stable eigenvalues.

5. The model solution expressed via Hypergeometric functions

We are now going to show that the model’s solution can be written in closed-form through the Hypergeometric function \(\text{Hypergeometric}_2F_1\) (see Appendix).

Lemma 3.

\[
k_t = a - b + be^{at} \left[ k_0^{1-\alpha} + (1 - \alpha) s \int_0^t \left( \frac{ae^{(\delta+g+at)}t}{a - b + be^{at}} \right)^{1-\alpha} dt \right],
\]

\[
L_t = \frac{ae^{at}}{a - b + be^{at}}.
\]

Proof. The logistic equation (5) can be solved by the method of separation of variables. The statement is immediate separating the \(L_t\) and \(t\) dependent parts of equation (5) and integrating both sides. The first equation of (6) is a Bernoulli differential equation, whose solution is known to be found by taking the substitution \(z = k^{1-\alpha}\). This yields \(\dot{z} = (1 - \alpha) s (1 - \alpha) (\delta + g + a - bL) z\), a linear differential equation with solution

\[
z_t = e^{-\int_0^t (1 - \alpha)(\delta + g + a - bL) dt} \left( z_0 + \int_0^t (1 - \alpha) s e^{\int_0^t (1 - \alpha)(\delta + g + a - bL) dt} dt \right).
\]
Since
\[ e^{-(1-\alpha)L_t}e^{(\delta+g+a-bL_t)}dt = e^{-(1-\alpha)L_t}e^{(\delta+g)L_t}dt = e^{-(1-\alpha)(\delta+g)t}L_t^{1-\alpha}, \]
equation (11) becomes
\[ z_t = e^{-(1-\alpha)(\delta+g)t}L_t^{1-\alpha}\left(z_0 + (1-\alpha)s\int_0^t e^{(1-\alpha)(\delta+g)t}L_t^{1-\alpha}dt\right). \]
The statement follows by rewriting this equation in terms of \( k_t \) and then substituting \( L_t \) given by (10).

\[ \square \]

**Remark 4.** \( L_t \) is a monotone increasing function from 1 to \( L_\infty = a/b. \)

The next result confirms analytically the fact that the economy converges to the steady-state no matter where the initial point is.

**Corollary 1.** \( \lim_{t \to \infty} (k_t, L_t) = (k_*, L_*). \)

**Proof.** Since it is \( (a - b + be^{at})/ae^{(\delta+g+a)t} \to 0 \), and, using \( L_t \geq 1 \), we obtain
\[ \int_0^t \frac{ae^{(\delta+g+a)t}}{a - b + be^{at}} \, dt = \frac{e^{(1-\alpha)(\delta+g)t} - 1}{(1-\alpha)(\delta + g)} \to +\infty, \]
the statement can be derived as an application of Hopital rule. In fact, we have
\[ \lim_{t \to \infty} k_t^{1-\alpha} = \lim_{t \to \infty} \frac{k_t^{1-\alpha} + (1-\alpha)s\int_0^t \frac{ae^{(\delta+g+a)t}}{a - b + be^{at}} \, dt}{a - b + be^{at}} = \frac{s}{\delta + g}. \]

\[ \square \]

**Proposition 2.** Let \( _2F_1 \) be the Hypergeometric function. Set \( \gamma = (1-\alpha)(\delta+g+a) \), \( M = b/(b-a) \). Then
\[ k_t = \frac{1 - Me^{at}}{(1-M)e^{\gamma/(1-\alpha)t}} \left( k_0^{1-\alpha} + \frac{a(1-\alpha)(1-M)^{1-\alpha}}{\gamma} \right) \cdot \left[ e^{\gamma t} _2F_1 \left( \frac{\gamma}{a}, 1-a, \frac{\gamma}{a} + 1; Me^{at} \right) - _2F_1 \left( \frac{\gamma}{a}, 1-a, \frac{\gamma}{a} + 1; M \right) \right]^{1-\alpha}. \]

**Proof.** Expressing \( L_t = (1-M)e^{at}/(1-Me^{at}) \), and operating the change of variable \( x = e^{at} \), we can write
\[ \int_0^t \frac{ae^{(\delta+g+a)t}}{a - b + be^{at}} \, dt = \frac{(1-M)^{1-\alpha}}{a} \int_1^{e^{at}} x^{\frac{1}{\gamma}-1} (1-Mx)^{-(1-\alpha)} \, dx. \]
The statement follows from (9) noting that
\[ \int_1^{e^{at}} x^{\frac{1}{\gamma}-1} (1-Mx)^{-(1-\alpha)} \, dx = \int_0^{e^{at}} x^{\frac{1}{\gamma}-1} (1-Mx)^{-(1-\alpha)} \, dx - \int_0^1 x^{\frac{1}{\gamma}-1} (1-Mx)^{-(1-\alpha)} \, dx, \]
\[ = \frac{a}{\gamma} e^{\gamma t} _2F_1 \left( \frac{\gamma}{a}, 1-a, \frac{\gamma}{a} + 1; Me^{at} \right) - \frac{a}{\gamma} _2F_1 \left( \frac{\gamma}{a}, 1-a, \frac{\gamma}{a} + 1; M \right), \]
having used the integral representation of an Hypergeometric function $2F_1$, the change of variable $r = e^{-at}x$ in the first integral, and the following two properties of the $\Gamma$-function: $\Gamma(1) = 1$ and $\Gamma(v + 1) = v\Gamma(v)$, for all $v > 0$. □

6. Conclusions

The Solow growth model assumes that labor force grows exponentially, which is not a realistic assumption because exponential growth implies that population increases to infinity as time tends to infinity. In this paper we propose to replace the exponential population growth with a more realistic equation, the logistic population growth law. This set up leads the model to be described by a two dimensional dynamical system. Within this framework, we have investigated the model’s dynamic and derived that the economy has a unique non-trivial steady state equilibrium, which is a node. Moreover, we have proved the model’s solution to be globally asymptotically stable and solvable in closed-form in terms of Hypergeometric functions.

7. Appendix

Let us briefly recall some facts about Hypergeometric functions (see Abramowitz and Stegun, 1972). The Gauss Hypergeometric function $2F_1(c_1, c_2, c_3; z)$, with complex arguments $c_1, c_2, c_3$, and $z$, is given by the series

$$2F_1(c_1, c_2, c_3; z) = \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_2)} \sum_{m=1}^{\infty} \frac{\Gamma(c_1 + m)\Gamma(c_2 + m)}{\Gamma(c_3 + m)} \frac{z^m}{m!},$$

where $\Gamma(\cdot)$ is the special function Gamma. The above series is convergent for any $c_1, c_2$ and $c_3$ if $|z| < 1$, and for any $c_1, c_2$ and $c_3$ such that $Re(c_1 + c_2 - c_3) < 0$ if $|z| = 1$. Fortunately, there are many continuation formulas of the Gamma Hypergeometric function outside the unit circle. The most practical continuation formulas consist in the integral representations of the Gamma Hypergeometric function. We shall use the following formula

$$2F_1(c_1, c_2, c_3; z) = \frac{\Gamma(c_3)}{\Gamma(c_1)\Gamma(c_3 - c_1)} \int_0^1 t^{c_1 - 1}(1 - t)^{c_3 - c_1 - 1}(1 - zt)^{-c_2}dt,$$

where $Re(c_1) > 0$, $Re(c_3 - c_1) > 0$, commonly known as the Euler integral representation.

References


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